Estimation of interpolation errors in scalp topographic mapping

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Abstract

Topographic maps are commonly constructed from electrical scalp recordings (such as EEGs and ERPs) using several different interpolation methods. It is important to determine the accuracy of such maps. Previous assessments of interpolation methods have been based on global error measures and the visual appearance of the topographic maps. However, the relationship of interpolation error to local contributing factors requires a more detailed analysis. In this paper, we use simulations to explore and quantify the relationship of error to global and local factors for different interpolation methods. We find that among the best interpolation methods, adequate electrode density is more important than the method used. For shallow sources, we show that local interpolation error is most correlated with potential gradient, and has a lesser correlation with distance to nearest electrode. The greatest correlation, however, is with the product of gradient and distance. Thus, interpolation error can be controlled locally by making the interelectrode distance inversely proportional to the expected potential gradient. With shallow sources, areas far from any electrode and having high apparent gradient are likely to have high interpolation error. Moreover, all areas far from any electrode may contain high interpolation errors, and should be interpreted with caution.

Keywords: Interpolation errors; Scalp topographic mapping

1. Introduction

Studies of the topographic patterns of electric fields (EEGs and ERPs) have become increasingly popular for investigating sensory, cognitive and motor activity in the human brain. To construct these topographic maps, various interpolation methods are used to estimate the electric potential values at scalp locations between actual recording sites. Commonly used interpolation methods belong to two different classes, the spline and nearest neighbor methods.

Informally, all methods interpolate by computing a function to describe the variations of voltage or current across the scalp, based on the known values of these quantities at points sampled by electrodes. For thin plate splines, this can be thought of as ‘bending’ an infinite flat sheet so that it passes through each of the given data values (represented as distances above or below the \(xy\)-plane at the recording sites) with as little bending effort as possible (Meinguet, 1979). Similarly, spherical splines use minimal bending energy in deforming a spherical surface to pass through the data values (represented as distances above or below the sphere at the recording sites) (Wahba, 1981). In contrast, nearest neighbor methods compute the value at a scalp location as the weighted average of data values from a specified number of nearest recording sites (Shepard, 1968).

Since interpolation provides estimates of potentials at scalp locations where no recording has been made, it is important to be able to estimate the accuracy of the maps obtained. The question of interest is how confident can we be that interpolated topographic features are real, instead of artifacts of the interpolation? For the spline and nearest neighbor interpolation methods, it is known mathematically that interpolation error decreases as a power of the greatest distance of a head point to its nearest electrode site (Duchon, 1978; Perrin et al., 1987; Perrin et al., 1989). Thus, with a dense enough electrode set, any of these methods would yield accurate topographical...
maps. Unfortunately, this relation does not provide numerical bounds on the error for specific interpolation problems, or prescribe the density required for acceptably accurate interpolation.

These issues have, however, been addressed in previous research. For example, some quantitative estimates of necessary electrode density have been made. Perrin et al. (1990) suggested that distance between electrodes should be less than half the shortest spatial period of the potential. Gevins (1990) provided an example using the N170 transient visual ERP component. The N170 had substantial energy at 4 cm/cycle. To adequately sample it, he concluded that electrodes placed 2 cm apart, or more than 128 electrodes over the head, would be required. Nunez and Westdorp (1994) showed that surface laplacians, obtained from spline functions interpolated over 64 electrodes, could pick out clumped sources with what they described as 'reasonable accuracy.' They reported even better results with a montage of 117 electrodes.

Several recent studies (Perrin et al., 1987, 1989; Soufflet et al., 1991; Soong et al., 1993) have compared the behavior of different interpolation methods using the qualitative appearance of topographic maps, as well as global statistical measures to give quantitative comparisons. A global measure is one pertaining to the topographic map as a whole: examples include the mean error, root mean square (RMS) error and various correlation coefficients.

Soong et al. (1993) compared thin plate and spherical spline methods, and nearest neighbor interpolation of EEGs for 31 electrode sites, over 1024 sample points in time. They collected error data by cross-validation, in which the potential at each of the sites, for each of the time slices, was in turn predicted by interpolation using the other 30 sites. They evaluated the error with four global measures: inaccuracy, precision, bias and tolerance. Inaccuracy was computed as the sum, over all electrode sites and all time slices, of the squared differences between true and interpolated EEG values, normalized by the sum of the squares of the true values; in other words, a normalized RMS measure. Precision and bias were each a type of correlation coefficient, respectively the correlation between true and interpolated values, and between true and error values, over each electrode and each time slice. Tolerance was the maximum squared error, normalized by the sum of the squares of the true values.

Soong et al. showed that the spline techniques performed better than nearest neighbor interpolation according to these measures, but no method performed acceptably with respect to inaccuracy and tolerance. The inaccuracy measures for nearest neighbor interpolation of diffuse brain activity indicated expected error averaging around 30%. For splines, the expected error was around 25%. Maximum errors were close to 70% and 50%, respectively. For focused epileptiform activity, expected errors for splines averaged close to 50%, with maximum error over 100%. The figures for nearest neighbor were higher. Two resulting conclusions were that 'systematic errors in topographic maps tend to be concentrated in the higher spatial frequencies,' and that the 'standard electrode density' commonly used clinically (the International 10-20 System) is inadequate for acceptable interpolation.

Soufflet et al. (1991) compared various three-dimensional interpolation methods and the spherical method for EEG mapping. The three-dimensional methods consisted of nearest neighbor and thin plate spline interpolation adapted to data points located in three-space, rather than on the plane or the sphere. Performance of each method was measured by comparing real data from 64 electrodes to data interpolated from a subsampled set of 28 electrodes. The resulting errors were evaluated using relative maximum error, extremal localization, and RMS error. Relative maximum was the ratio of the maximum interpolated value to the maximum true value. The error of extremal localization was the angle (subtended at the point in the head equidistant from inion and nasion) between maximum true and interpolated values in the larger 64-electrode set. The RMS error used here was the same as the square root of the inaccuracy measure by Soong et al. (1993). According to this measure, the expected error for spherical splines was approximately 20%; slightly lower but still consistent with the results obtained by Soong et al. The authors also made visual comparisons between topographic maps. They concluded that three-dimensional thin plate splines gave better spatial resolution than planar interpolation (such as spherical splines).

Perrin et al. (1987, 1989) performed simulation studies of interpolation over electrode sets of 19 and 41 locations, each containing the International 10-20 System locations. They evaluated the different interpolation methods by visually comparing interpolated topographic maps with true forward solution maps, and also by calculating RMS error, maximum error and degree error of extremal localization. The data consisted of the differences between interpolated and true potentials at each of 5783 points located on the upper hemisphere of the spherical head model. The authors reported that spherical splines seemed marginally better than the best thin plate method (Perrin et al., 1989) in regions not well covered by electrodes, and that thin plate splines were superior to nearest neighbor methods (Perrin et al., 1987).

In sum, it is known that high spatial frequencies of potential require dense electrode arrays for adequate interpolation by any method; and for relatively low numbers of electrodes, there is a difference in interpolation methods. When low electrode densities are used, there is a question as to whether any interpolation method is reliable. Nevertheless, there has been no detailed quantification of either the relationship of error to spatial frequency and electrode density, or the relationship of interpolation...
methods to one another over a wide range of electrode sets and potentials.

The goal of the present study is to further quantify these relationships, by considering both global and local characteristics of the electrode set and potential. The global characteristics we examined were number of electrodes $N$ and source eccentricity $E$ (eccentricity is the percentage distance of a dipole source from the center of the head). We also used the following local characteristics, which pertain not to the entire head but only to a particular point: the distance $d$ of that point to the nearest electrode in the electrode set, the gradient $g$ of the true potential and the curvature $c$ of the true potential surface. Here gradient means the greatest rate of change of the potential surface at a given point: the greatest 'slope' of the surface. We defined curvature as the absolute value of the average rate of bending of the surface at a point. It was calculated as the distance of the given point on the potential surface from the average of all its neighboring points on the surface, divided by the maximum separation of any two of the neighbors.

These concepts are illustrated schematically in Fig. 1. The parameters $d$ and $c$ for the head point $H_0$ are labeled in the figure. The distance $d$ is the distance on the $X$-axis from $H_0$ to the nearest electrode $e$. Curvature $c$ is the vertical distance between the potential value at $H_0$ and the average of the values at all neighboring points, which are represented by $H_1$ and $H_2$. Gradient is calculated as $g = |a/b|$, where this ratio is the maximum over all neighboring points of $H_0$, represented here by $H_2$.

We are interested in answering the following questions:
(a) What are the best and worst interpolation methods?
(b) Does the choice of method depend upon global and local factors?
(c) How does error interact with the global and local factors?
(d) How big can interpolation errors be, and under what conditions?
(e) Are there some useful rules of thumb for predicting and mitigating interpolation errors?

2. Methods

2.1. Interpolation methods

We tested three orders of thin plate spline, as well as spherical spline and nearest neighbor interpolation methods.

Thin plate splines are defined on the $xy$-plane. Let $(x, y)$ be planar coordinates. For each value of $m = 2, 3, 4$, define the functions

$$f_{m-1}(x, y) = (x^2 + y^2)^{m-1} \frac{\log (x^2 + y^2 + 0.001)}{0.001}$$

and

$$g_{m-1}(x, y) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \frac{c_{ij} x^{i-j} y^j}{i+j}$$

The quantity $0.001$ on the right side of the equation defining $f_{m-1}$ causes the function to be infinitely differentiable at all points on the scalp surface, including when $x = y = 0$ (Harder and Desmarais, 1972). Such an added constant models a distributed potential -- an electrode -- at the point $x = y = 0$. In terms of the natural unit of distance in our experiments (the radius of the spherical head model), the size of this constant would be about 0.03 to represent a real electrode. For our modeling studies, we chose the smaller value to represent an idealized point electrode. In practice, the differences in the resulting function values were negligible over the range of the head.

Let the $N$ data locations be the points $(x_k, y_k)$, with data values $v_k$, for $k = 1, \ldots, N$. Then the thin plate spline of order $m$ is given by

$$F_m(x, y) = g_{m-1}(x, y) + \sum_{k=1}^{N} a_k f_{m-1}(x - x_k, y - y_k)$$

This formula was implemented on the sphere by projecting the sphere on to the $xy$-plane, using the stereographic projection which maps the south pole to the point at infinity. Thus the quantity $f_{m-1}(x - x_k, y - y_k)$ is a function of the (projected) squared distance from the point with projected coordinates $(x, y)$ to the projected coordinates of the $k$th data point. From the formula for $f_{m-1}(x, y)$, it is clear that $f_{m-1}(x - x_k, y - y_k)$ vanishes at the $k$th data point.
The interpolation problem is solved by finding specific values for the coefficients $a_k$ and $c_{ij}$ in the definition of $F_m$. These values are determined by requiring that $F_m(x_k, y_k) = v_k$ at each of the data locations, and also that the coefficients $a_k$ satisfy an orthogonality relation with a specific matrix derived from the data point coordinates (see Perrin et al., 1987 for further details).

Spherical splines are defined on the surface of a sphere (Wahba, 1981). Let $s, t$ be two points on the sphere, and $\cos(s, t)$ be the cosine of the angle between $s$ and $t$. Define

$$q_m(s, t) = \frac{1}{4\pi} \sum_{k=1}^{N} \frac{2k + 1}{k^m (k + 1)^m} P_k(\cos(s, t))$$

where $P_k$ is the $k$th degree Legendre polynomial. For electrode locations $s_k$ on the sphere having potential values $v_k$, the spherical spline of order $m$ is given by

$$G_m(s) = b_0 + \sum_{k=1}^{N} b_k q_m(s, s_k)$$

Interpolation involves solving for the coefficients $b_k$, again by requiring $G_m(s_k) = v_k$ at all data locations, and making the coefficients satisfy a certain orthogonality relation. Perrin et al. (1989) found that $m = 4$ gave the best results, and we use this value.

In contrast to the spline methods, which use all the data points, the nearest neighbor methods (Shepard, 1968) use only a specified number of the data points nearest to any location. Let $v_k$ be the data value at point $p_k$, for $k = 1, \ldots, N$. Let $x$ be the point to be interpolated, and $d_k$ be the distance of $x$ to $p_k$. Let $M$ be the number of nearest neighbors specified. Then

$$H_m(x) = \frac{\sum_{k=1}^{M} v_k / d_k^m}{\sum_{k=1}^{M} 1 / d_k^m}$$

if $x$ is not a data point $p_k$, and $H_m(p_k) = v_k$. The distance exponent $m$ was set to 2 in our study. The number of neighbors $M$ was set to 4. This is the number used by Soong et al. (1993) and Perrin et al. (1987).

### 2.2. Generating the model data

Our simulations involved placing multiple dipole point sources, of randomly determined tangential or radial moment with unit magnitude (expressed in microvolts), at locations within the spherical head model, and then generating the surface potentials, or forward solution, at the surface of the sphere. Each forward solution was the sum of the potentials due to each of the dipoles.

We used the three-shell model (Nunez, 1981; Nunez and Westdorp, 1994), consisting of concentric spheres having radii of 0.888, 0.945 and 1.00, to represent the surface of the brain, the surface of the skull, and the scalp surface, respectively. The radius of our spherical head model was 1, and throughout this article, all units of distance are implicitly expressed in terms of head radii. The impedance of brain and scalp were assumed to be equal, and that of the skull to be 80 times greater (Nunez, 1981).

We generated these potential values at 5196 locations placed over the upper part of the sphere above a latitude of 100 degrees from the north pole (the north pole is defined as Cz of the International 10-20 System); this was our reference set. This number of points was similar to the 5783 points used over the upper hemisphere in Perrin et al. (1987).

We also generated forward solution values at four additional sets of locations, which we used as our electrode sets, having 26, 56, 91 and 151 evenly spaced electrodes. These locations were separate from the 5196 points of the reference set. We created the electrode sets by evenly subdividing the faces of an icosahedron (a 20-face regular polyhedron) and projecting the resulting vertices onto the sphere, giving approximately evenly spaced electrode locations. We chose this range of numbers to approximate the range cited in the literature; from the 28 electrodes used by Soufflet et al. (1991) and the 19 and 41 electrodes used by Perrin et al. (1987, 1989), to the 128 electrodes cited by Gevins (1990).

Not all of the electrode sets completely contained the portion of the sphere covered by the reference set. In fact, the 26- and 91-electrode sets extended down only to 90° from the north pole; the 151-electrode set extended to 96°. For these electrode sets, we ignored any data generated at points outside the covered area. In other words, we analyzed only errors due to interpolation (generated within the area of an electrode set) rather than extrapolation (generated outside of an electrode set).

To gather error data for a given interpolation method, we first generated the forward solutions for a dipole source file onto a given electrode set and also onto the reference set. We then used the forward solution values at the electrode set locations to interpolate values at the locations of the reference set. The absolute value of the difference between true and interpolated values at each reference set point was the error datum at that point. We paired each error datum with the global and local parameter values at the same point.

Our dipole source files each had five dipoles with random longitude and latitude coordinates, randomly determined radial or tangential orientation, and fixed eccentricity. We used this arrangement of five sources firing simultaneously, having the same eccentricity but randomly differing in location and orientation, to create a summed potential at the surface of the sphere corresponding to each source file. Our purpose was to obtain better simulations of actual ERP potentials than we could get.
from using source files consisting of only one dipole. Our dipole moments were all set at $1 \mu\text{V}/(\text{head radius})$, and all other potentials, including interpolation errors, are thus expressed in terms of microvolts.

To compute the mean and maximum global errors over all electrode sets and all source eccentricities, we used forward solutions of six dipole source files for each eccentricity value of 20, 40, 60, 80 and 88% and each of the four electrode sets. To compute the local mean errors as functions of the local parameters, we used the same methods as for global errors except that we used 50 dipole source files instead of six, at a fixed source eccentricity and electrode set. Mean and maximum global errors and local mean error are defined in the next paragraph.

2.3. Statistical analysis

We computed both global and local statistical measures. Global mean error was calculated as the average absolute value interpolation error over all 5196 reference set points and all simulations. We also computed global maximum error over all points in the reference set and all simulations, and average maximum error as the average of the maximum errors in each simulation. Let $N$ be the number of simulations and $S_x$ be the $k$th simulation. Let $T_{x_k}$ and $I_{x_k}$ be the true and interpolated values, respectively, at the $n$th point of the reference set during the $k$th simulation. Then

$$
\text{Global mean error} = \frac{\sum_{k=1}^{N} \sum_{n=1}^{5196} |T_{x_k} - I_{x_k}|}{5196N}$$

$$
\text{Average max error} = \text{mean}\left\{\max_{n}|T_{x_n} - I_{x_n}|\right\}
$$

and

$$
\text{Global max error} = \max_{k,n}|T_{x_k} - I_{x_k}|
$$

Global maximum error is the greatest error recorded over all points in all simulations (the ‘worst of the worst’). It is thus the worst error observed in the entire set of experiments. Average maximum error is the worst error that is likely to occur, on average, in a simulation (the ‘mean worst’).

Local mean error was calculated as the average absolute value error over all points having a value of a parameter or product of parameters within a small range of a specified value. For example, if we wished to test for a functional relationship between error and $d$, the distance to nearest electrode, we grouped all data points by their $d$ values. Specifying a value of $d_0$ of $d$, we computed the mean absolute value error of all points having $d$ within a small range of $d_0$. Let $F(d_0)$ be the local mean error when $d_0 - \varepsilon \leq d \leq d_0 + \varepsilon$. Let $N_{d_0}$ be the number of points in the set of simulations for which $d$ satisfies this inequality. Then

$$
F(d_0) = \sum_{d_0 - \varepsilon \leq d \leq d_0 + \varepsilon} \frac{|T_n - I_n|}{N_{d_0}}
$$

Thus, local mean error was local in the sense that at a head point, the measured parameter value yielded an average error based on the value of the parameter at that point. We graphed the mean error as a function of the local parameter values.

We also computed the sample correlation coefficients between absolute error and all the parameters. The correlation coefficient $C(X,Y)$ between two variables $X,Y$ measures the degree to which $X$ and $Y$ are linearly related (Weselowsky, 1976).

When computing the correlations between error and each of the local parameters $d, g$ and $c$ or their products, our sample space consisted of error and parameter values for each of the head points inside the electrode set and each of the 50 source files, at each fixed source eccentricity, electrode set and interpolation method. While the number of head points varied depending on the area covered by the electrode set, it was never less than 4800 points, giving us at least $50 \times 4800 = 240 000$ sample values to compute each sample correlation coefficient. To compute the correlations between error and the global parameters $E$ and $N$ or products of global and local parameters, our sample space was drawn from six different source files and allowed $E$ and $N$ to vary over their ranges. This gave us at least $4 \times 5 \times 6 \times 4800 = 576 000$ instances of each error-parameter pair. In either case, the large size of our sample space ensured that our sample correlations were highly reliable estimates of the true correlations.

The correlation coefficient assumes values from $-1$ to 1, with $C(X,Y) = \pm 1$ indicating a perfect linear relationship, where the slope is positive (+1) or negative (−1). Values of $C(X,Y)$ ≥ 0.5 or $C(X,Y)$ ≤ −0.5 indicate moderately strong positive or negative linear correlations (Younger, 1979, p. 244). $C(X,Y) = 0$ indicates no linear correlation. Because of our large number of sample data points, any value of $|C(X,Y)| > 0.01$ is a virtually certain indication (above 99.9% probability) that there exists a correlation between $X$ and $Y$. 

1 Errors close to the global maximum in magnitude generally occurred at a only a small percentage of the points out of all the simulations. However, when viewed in conjunction with the mean errors and expected maximum errors, global maximum errors are useful to show how bad the error can be. These errors could occur at locations where true maxima or minima were greatly under- or overestimated by the interpolation method, and also at places where the method showed an extreme value that was not really there. In particular, thin plate 4 interpolation produced topographic maps often having wild oscillations that did not reflect the true potential surfaces. This resulted in particularly high global maximum errors.
3. Results

3.1. Comparison of interpolation methods using global measures

We compared interpolation methods both by computing the global mean absolute error for each method (averaged over all reference set points and simulations) and maximum error (maximum over all reference set points and simulations) and also by graphing the local mean error. We are reporting average as well as maximum error to give a more complete picture of the performance of the interpolation methods, and to provide a basis of comparison with the inaccuracy and tolerance measures of Soong et al. (1993).

3.1.1. Overall global error

Fig. 2 shows global mean error for each interpolation method, computed over all electrode sets and all eccentricities. Fig. 3 shows global maximum error for each interpolation method. Thin plate 2 has the lowest mean error by a slight margin, while spherical has the lowest maximum error. Thin plate 4 shows a very large maximum error. The error magnitudes shown are absolute, rather than normalized relative to the true potential magnitudes. Thus, in order to interpret the data, it is useful to note that the maxima of true potential forward solutions were roughly 1.5 μV at 88% source eccentricities. Forward solutions of sources at this eccentricity defined our worst case evaluation of the interpolation methods.

According to Fig. 3, it follows that the global maximum errors for all methods are a high percentage of the true maxima. All of these errors occurred from interpolation of data from sources at 88% eccentricity. They are about 66% (of 1.5 μV) for spherical and thin plate 2 splines, over 100% for thin plate 3 and nearest neighbor, and well over 200% for thin plate 4. These percentages are of the same order of magnitude as the tolerance measures reported for nearest neighbor, spherical and thin plate splines in Soong et al. (1993, Table IV). The average maximum errors, on the other hand, are lower, ranging from slightly more than 11% (in spherical splines) to almost 28% (in thin plate 4) of the true maxima. Finally, the mean errors are quite low relative to these maxima.

When error is taken over all ranges of electrode sets and eccentricities, the results suggest that the best method is thin plate 2 or spherical spline. Thin plate 3 is slightly worse, while thin plate 4 and nearest neighbor are markedly worse. We now compare the behavior of the methods as a function of different eccentricities and electrode sets, to see whether the relationship among the methods changes for particular values of these parameters.

3.1.2. Global error versus source eccentricity

Fig. 4 plots global mean interpolation error over all electrode sets as a function of source eccentricity. In Fig. 4, interpolation error for each eccentricity value is averaged over the four electrode sets. For all methods, error increases roughly exponentially with eccentricity. At low eccentricities (deep sources), the spherical spline has the lowest error by a wide margin, more than a factor of 10 better than the next best, which are thin plate 3 or 4. At high eccentricities (shallow sources), all errors are relatively close together, differing by no more than a factor of 3. At 80 and 88% eccentricities, the lowest error belongs to thin plate 2. Nearest neighbor is high throughout, but increases less steeply than the other methods and is actually lower than thin plate 4 at 88% eccentricity.

3.1.3. Global error versus number of electrodes

Fig. 5 plots global mean interpolation error taken over all electrode sets as a function of number of electrodes. In Fig. 5, interpolation error for each eccentricity value is averaged over the four electrode sets. For all methods, error increases roughly exponentially with number of electrodes. At low eccentricities (deep sources), the spherical spline has the lowest error by a wide margin, more than a factor of 10 better than the next best, which are thin plate 3 or 4. At high eccentricities (shallow sources), all errors are relatively close together, differing by no more than a factor of 3. At 80 and 88% eccentricities, the lowest error belongs to thin plate 2. Nearest neighbor is high throughout, but increases less steeply than the other methods and is actually lower than thin plate 4 at 88% eccentricity.

Fig. 2. Global mean interpolation errors for each method, taken over all electrode sets and all eccentricities, with six simulations from random sources at each eccentricity.

Fig. 3. Global maximum and average maximum interpolation errors for each method, taken over all electrode sets and all eccentricities, with six simulations from random sources at each eccentricity.

Fig. 4. Global mean interpolation error over all electrode sets as a function of source eccentricity. In Fig. 4, interpolation error for each eccentricity value is averaged over the four electrode sets. For all methods, error increases roughly exponentially with eccentricity. At low eccentricities (deep sources), the spherical spline has the lowest error by a wide margin, more than a factor of 10 better than the next best, which are thin plate 3 or 4. At high eccentricities (shallow sources), all errors are relatively close together, differing by no more than a factor of 3. At 80 and 88% eccentricities, the lowest error belongs to thin plate 2. Nearest neighbor is high throughout, but increases less steeply than the other methods and is actually lower than thin plate 4 at 88% eccentricity.

Fig. 5. Global mean interpolation error taken over all electrode sets as a function of number of electrodes. In this figure, interpolation error for each electrode set is averaged over all source eccentricities.
shows error declining exponentially with increasing number of electrodes. Figs. 4 and 5 together suggest that the eccentricity of a set of sources over its reasonable possible range affects the mean amplitude error more than does the number of electrodes in a range from 26 to 151. Overall, thin plate 2 shows the lowest error, with spherical spline being slightly higher. Error for thin plate 4 is uniformly high. Error for nearest neighbor is the highest, by a factor of up to 10 times thin plate 2.

The errors displayed in Figs. 2–5 suggest that spherical spline or thin plate 2 are the best interpolation methods, while nearest neighbor is the worst. Thin plate 4 also has error that is generally high. Error for nearest neighbor is the highest, by a factor of up to 10 times thin plate 2.

3.2. Contribution of global versus local parameters

Next we calculated the correlations between local mean error and each of the five parameters, as well as products of these parameters.

3.2.1. Overall correlations between local mean error and parameters

Table 1 shows the correlation between the local mean error for each interpolation method and each of the five parameters: eccentricity $E$, number of electrodes $N$, distance to head point $d$, gradient $g$ and curvature $c$. It also shows correlations between error and various products of these parameters.

We examined products of parameters as representing the interaction of two or more conditions. Within the range of parameter values we encountered, higher product values resulted from strong contributions of all conditions in the product. A low value for one or more of the conditions produced a lower value for the product. We wished to understand how conditions interact to produce interpolation error. Do the global factors $(E,N)$ interact with the local $(d,g,c)$? Do parameters associated with the sources $(E,g,c)$ interact with the parameters associated with the electrodes $(N,d)$?

Table 1 shows that for all methods, the single parameter having the largest correlation with error is the gradient $g$. For spherical spline, thin plate 2 and nearest neighbor, this correlation is in the moderately strong range. At the other extreme, curvature $c$ appears to be virtually uncorrelated with error. The global factors $E$ and $N$, and the local factor $d$, show weak correlations.

The largest correlations with error, however, belong to products of parameters (for each method, the largest correlation is in boldface). For all but nearest neighbor, $Edg$ has the highest correlation to error, and in this case it is close to the highest. For all methods except thin plate 4, the correlation of error with $Edg$ is in the moderately strong range.

3.2.2. Local mean error versus $Edg$

Table 1 indicates that at each head point, $Edg$ is the best or close to the best predictor of local error. Fig. 6 plots how local mean error varies as a function of $Edg$ for each interpolation method. It shows that for all interpolation methods, local mean error is a roughly linear function of $Edg$. Thin plate 2 has the lowest error at low to moderate values of $Edg$ and spherical has the lowest at the higher values of $Edg$. Spherical and thin plates 2 and 3 are fairly close, while nearest neighbor has the highest error at all ranges of $Edg$.

Having found that $Edg$ is generally the best predictor...
of error for all eccentricities and interpolation sets, we next examine how variation of $E$ and $N$ may influence the error contributions of the local parameters $d$, $g$, and $c$. Hence forth we focus only on error produced by spherical or thin plate 2 splines, which are the best methods as determined by the global error measures seen in Figs. 2–5, and by the local error measure as seen in Fig. 6.

3.2.3. Correlations with local parameters at fixed $E$

Table 2 shows the correlations of interpolation error with local parameters, at low and high eccentricities ($E = 0.20, 0.88$). For each method, the error correlations of all but $d$ are low at 20% eccentricity. The low correlations are due to the mean error being close to 0. Thus, no parameter is a good predictor of error. At 88%, gradient has the highest correlation of any single parameter, while $d$ has declined both relatively and absolutely. Nevertheless, the highest correlation belongs to $dg$. Curvature has about the same correlation as $d$ at high eccentricity.

3.2.4. Correlations with local parameters at fixed $N$

Table 3 shows correlations of error with parameters at low and high numbers of electrodes ($N = 26, 151$). The lower overall magnitudes of the correlations compared with Table 2 are the result of averaging over all source eccentricities. A comparison of Tables 2 and 3 shows that changing the value of $N$ (from 26 to 151) produces smaller changes in the correlation coefficients than changing the value of $E$ (from 20 to 88%). There is a decline in all coefficients going from 26 to 151 electrodes, but smaller than the increase in the coefficients (except for $d$) going from low to high eccentricity. However, both tables show the clear pattern that correlation of parameters with error is highest when error is highest (i.e. at low $N$ and at high $E$), and lower with lower errors.

3.2.5. Local mean error versus $dg$

In Fig. 7 we now plot local mean error of thin plate 2 versus $dg$ for each electrode set, with sources at 88% eccentricity. The graph shows a roughly linear relationship between error and $dg$ on each electrode set. Fig. 8 plots error for each source eccentricity, from interpolation over 26 electrodes. Here the error declines sharply with decreasing eccentricity, until it is barely visible at 20%. The maximum values of $dg$ decline more quickly also, with the decrease in gradient as eccentricity decreases.

To summarize these results, for thin plate 2 or spheri-

---

### Table 1

Correlation of absolute interpolation error with 5 parameters and products of parameters (highest correlations with error for each interpolation method are in boldface)

<table>
<thead>
<tr>
<th></th>
<th>ss</th>
<th>tp2</th>
<th>tp3</th>
<th>tp4</th>
<th>nn</th>
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<tbody>
<tr>
<td>$E$</td>
<td>0.31</td>
<td>0.26</td>
<td>0.29</td>
<td>0.28</td>
<td>0.23</td>
</tr>
<tr>
<td>$N$</td>
<td>−0.19</td>
<td>−0.19</td>
<td>−0.20</td>
<td>−0.17</td>
<td>−0.24</td>
</tr>
<tr>
<td>$d$</td>
<td>0.21</td>
<td>0.21</td>
<td>0.23</td>
<td>0.21</td>
<td>0.29</td>
</tr>
<tr>
<td>$g$</td>
<td>0.51</td>
<td>0.55</td>
<td>0.46</td>
<td>0.30</td>
<td>0.55</td>
</tr>
<tr>
<td>$c$</td>
<td>0.02</td>
<td>0.04</td>
<td>0.02</td>
<td>−0.01</td>
<td>0.15</td>
</tr>
<tr>
<td>$dg$</td>
<td>0.63</td>
<td>0.67</td>
<td>0.58</td>
<td>0.40</td>
<td>0.66</td>
</tr>
<tr>
<td>$Edg$</td>
<td>0.67</td>
<td>0.694</td>
<td>0.617</td>
<td>0.43</td>
<td>0.66</td>
</tr>
<tr>
<td>$dg/N$</td>
<td>0.65</td>
<td>0.691</td>
<td>0.613</td>
<td>0.426</td>
<td>0.62</td>
</tr>
<tr>
<td>$Edgc/N$</td>
<td>0.63</td>
<td>0.66</td>
<td>0.59</td>
<td>0.40</td>
<td>0.67</td>
</tr>
</tbody>
</table>

---

### Table 2

Correlations of error with local parameters for spherical and thin plate 2 splines, at low and high eccentricities, averaged over all electrode sets (highest correlations in each column are boldfaced)

<table>
<thead>
<tr>
<th></th>
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<th>tp3: 88%</th>
<th>tp4: 88%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
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<td>0.23</td>
<td>0.16</td>
<td>0.11</td>
</tr>
<tr>
<td>$g$</td>
<td>−0.04</td>
<td>−0.11</td>
<td>0.60</td>
<td>0.05</td>
</tr>
<tr>
<td>$c$</td>
<td>0.07</td>
<td>0.04</td>
<td>0.15</td>
<td>0.14</td>
</tr>
<tr>
<td>$dg$</td>
<td>0.10</td>
<td>0.07</td>
<td>0.63</td>
<td>0.67</td>
</tr>
<tr>
<td>$gc$</td>
<td>0.05</td>
<td>0.01</td>
<td>0.53</td>
<td>0.57</td>
</tr>
<tr>
<td>$dc$</td>
<td>0.05</td>
<td>−0.01</td>
<td>0.53</td>
<td>0.57</td>
</tr>
<tr>
<td>$dgc$</td>
<td>0.13</td>
<td>0.10</td>
<td>0.58</td>
<td>0.61</td>
</tr>
</tbody>
</table>

### Table 3

Correlations of error with local parameters for spherical and thin plate 2 splines, interpolating from 26 and 151 electrodes, averaged over high (88%) and low (20%) source eccentricities (highest correlation in each column is boldfaced)

<table>
<thead>
<tr>
<th></th>
<th>ss: 26</th>
<th>tp2: 26</th>
<th>ss: 151</th>
<th>tp2: 151</th>
</tr>
</thead>
<tbody>
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<td>$d$</td>
<td>0.19</td>
<td>0.16</td>
<td>0.11</td>
<td>0.09</td>
</tr>
<tr>
<td>$g$</td>
<td>0.27</td>
<td>0.20</td>
<td>0.30</td>
<td>0.28</td>
</tr>
<tr>
<td>$c$</td>
<td>0.17</td>
<td>0.15</td>
<td>0.09</td>
<td>0.05</td>
</tr>
<tr>
<td>$dg$</td>
<td>0.35</td>
<td>0.34</td>
<td>0.34</td>
<td>0.32</td>
</tr>
<tr>
<td>$dc$</td>
<td>0.25</td>
<td>0.22</td>
<td>0.14</td>
<td>0.09</td>
</tr>
<tr>
<td>$gc$</td>
<td>0.32</td>
<td>0.31</td>
<td>0.28</td>
<td>0.25</td>
</tr>
<tr>
<td>$dgc$</td>
<td><strong>0.38</strong></td>
<td><strong>0.37</strong></td>
<td>0.33</td>
<td>0.29</td>
</tr>
</tbody>
</table>
Fig. 7. Local mean error for thin plate 2 interpolation, plotted against $dg$ for each electrode set.

cal splines, the correlations of local parameters with error vary depending on the values of eccentricity and number of electrodes. The variation associated with $E$ is greater than that associated with $N$. All correlations except $d$ increase markedly going from low to high eccentricity. All correlations decline, by a smaller amount, going from low to high number of electrodes. The single parameter with the highest correlation in all instances except $E = 0.20$ is $g$; but either $dg$ or $dgc$ always has higher correlation with error.

Likewise, local mean error as a function of $dg$ varies with $E$ and $N$. In each case the function is roughly linear. It declines modestly with each increase of $N$. It declines more sharply with each decrease of $E$.

3.3. Local mean error and topographic maps

Next we focus on the relationship between our graphs of local mean error versus $dg$, and the visual errors occurring in topographic maps. The issue is the following: given our knowledge of the way in which expected error depends upon $dg$, how confident can we be that what we are seeing in a topographic map of recorded data is real, rather than an artifact? The following example provides some insights.

Fig. 9 plots thin plate 2 error for 88% eccentricity sources over 26 electrodes, with error bars at selected values of $dg$ indicating the percentage of errors below the tic mark levels shown. To predict error at a head point in a topographic map, we estimate the values of $d$ and $g$ at that point, then refer to Fig. 9 for an estimate of the error distribution associated with $dg$.

The values of $d$ and $g$ can be approximately estimated from the topographic map itself. It is easy to estimate $d$ by looking at the distance to the closest electrode on the topographic map. Gradient is, however, more difficult, because we must rely on the gradient of the interpolated potential as an estimator of the true potential gradient, and this maybe inaccurate. In addition, it is harder to visually estimate gradient than distance to head point. Since $g$ measures the greatest steepness of the potential surface at a point, it will be high at locations among closely spaced contour lines. If the point is far from any contour line, the gradient is low. Also, $g = 0$ at any extremum or saddle point.

We apply this method to the topographic map of simulated data in Fig. 10, produced by thin plate 2 interpolation over 26 electrodes. There were five sources at 88% eccentricity. We expect the highest error at head locations having high values of $dg$, that is, far from any electrode and having closely spaced contour lines. In Fig. 10, estimating true gradient from the interpolated gradient, we would expect high values of $dg$ in the upper right quadrant of the scalp, between the two apparent extrema.
Fig. 10. Topographic map interpolated by thin plate spline, from simulated 5-dipole source file at 88% eccentricity, over 26 electrodes.

where the contour lines are closely spaced and far from any electrode. Referring to Fig. 9, a value of $d_g = 1.5$ (close to the maximum) yields an expected local mean error of about 0.40 $\mu$V. The error has a 90% probability of being less than 0.75 $\mu$V. Looking at a second area of Fig. 10, the upper left quadrant of the top of the scalp, we see lower gradients indicated by more widely spaced contour lines. Here we estimate the value of $d_g$ to be less than half the maximum, even at points far from an electrode. From Fig. 9, a value of $d_g = 0.7$ yields an expected error of about 0.2 $\mu$V, with a 90% chance of being under 0.45 $\mu$V. Thus, we have estimates of expected error and error ranges for two regions of Fig. 10, as illustrations of our method.

We checked our predictions by looking at the true potential in Fig. 11. Our prediction of error in the upper right quadrant was accurate. Although the interpolation showed two extrema at roughly the correct magnitudes, it placed them too far apart. Comparing the value scales at the right of each of the Figs. 10 and 11 with the region in question shows that the actual error magnitude fell within the 90% value of 0.75 $\mu$V predicted by Fig. 9. On the other hand, our prediction for the upper left quadrant was inaccurate; we predicted moderate error, when in fact Fig. 11 shows a pair of extrema of high positive and negative values in the upper left quadrant. The interpolation error here was at least 0.80 $\mu$V in magnitude. The problem in this case was using the interpolated gradient as an indicator of the true gradient. This led to a large underestimate of the actual interpolation error. However, we note that the extrema shown in Fig. 11 are in fact far from any electrode shown in Fig. 10.

This example illustrates:

1. two of the ways that interpolation error produces visual artifacts in a topographic map: extrema not shown, and extrema incorrectly located.
2. the strength and limitation of error prediction from the topographic map itself.
3. when interpolating data from high eccentricity sources, interpolation at head locations sufficiently far from an electrode should always be considered suspect.

3.4. Mitigating error

Finally we show that high spatial frequencies of potential (associated with high eccentricity sources) can be resolved using locally dense electrode montages, if we have prior knowledge of the area where the activity is
likely to be. We illustrate this by resolving the two potential extrema in the upper left scalp quadrant of Fig. 11, using a set of 47 electrodes in which 24 of the electrodes are placed over the area of interest. Fig. 12 shows the thin plate 2 interpolation of the same five sources, using this electrode set: the 24 electrodes were placed at the same density as our 151 electrode set. Referring to Fig. 11, the resolution of the two extrema in this sector was indeed quite good.

4. Discussion

We have analyzed interpolation error with respect to global and local parameters. Our global parameters were source eccentricity $E$ and number of electrodes $N$. Local parameters were the distance of a given head point to the nearest electrode $d$, the potential gradient $g$, and the potential surface curvature $c$. By using model data, we addressed the way these factors influence error for different interpolation methods, and how the parameters are correlated with error. Several observations about interpolation error provide information useful in predicting when error is likely to be greatest, and how these errors may be minimized.

4.1. Differences among interpolation methods

Although the difference in error among interpolation methods varied with both local and global parameters, it is nonetheless possible to provide several generalizations. Thin plate 2 and spherical splines were the best according to our local and global error measures, with thin plate 3 very close to these in accuracy. Thin plate 2 had the lowest global mean error (Fig. 2), the lowest error averaged over all electrode sets at high source eccentricity (Fig. 4), the lowest error averaged over all eccentricities for each electrode set (Fig. 5), and the lowest error at all but the highest values of the parameter product $Edg$ (Fig. 6). Spherical spline had the lowest error by a large factor at low to moderate source eccentricities (Fig. 4) and by a small difference at high values of the parameter product $Edg$ (Fig. 6). These results suggest that it would be hard to choose which of these two methods is best overall. However, for relatively low numbers of electrodes (giving a high value of $d$), coupled with high source eccentricities and regions of high gradients, the spherical spline method is the best by a small margin of error (Fig. 6). With higher numbers of electrodes, resulting in lower values of $d$, thin plate 2 is actually the best method.

Our first conclusion, that spherical spline interpolation is the best method for low electrode densities, is consistent with that reached by Perrin et al. (1989). In that study, they stated that thin plate and spherical splines gave similar results in areas that are well covered by electrodes, while in sparsely covered areas, spherical splines seemed to have a slight advantage. This was based on a visual comparison of thin plate and spherical spline interpolation, using a single dipole source and 16 electrodes. Our second conclusion results from extending their comparison to interpolation over larger numbers of electrodes. In that situation, thin plate 2 is the best method.

In contrast to the spherical spline and thin plate 2 methods, nearest neighbor had the largest error of all methods with almost every measure, and thin plate 4 was also higher than the other splines. On this basis we conclude that nearest neighbor and thin plate 4 are not in the same class as the first three methods. This is consistent with conclusions reached by Perrin et al. (1987). Thus, nearest neighbor and thin plate 4 probably should not be used for interpolation if any of the other methods are available.

Regarding thin plate 2 and 3, our conclusions are consistent with the observation made by Soong et al. (1993) that there was little difference between the thin plate and spherical spline techniques. However, they found that increasing the order of thin plate interpolation gave worse measures of inaccuracy and precision, but better measures of bias and tolerance. For example, thin plate 2 had the lowest inaccuracy, but the highest tolerance measure. Thus, they chose thin plate 3 as having the best balance of error measures.

Our results showed that thin plate 2 had the lowest global mean error (Fig. 2), as well as the lowest error averaged over all dipole source eccentricities on each electrode set (which is roughly equivalent to the inaccuracy of Soong et al.). These results are consistent with the findings of Soong et al. However, we also found that among the thin plate methods, thin plate 2 had the lowest average maximum and global maximum errors (Fig. 3). In addition, for thin plate 2, local mean error as a function of $Edg$ was lower than other thin plate methods except at the highest values of $Edg$ (Fig. 6).

These reasons led us to choose thin plate 2, rather than thin plate 3, as the best thin plate method. We note, however, that our global maximum error measure is the worst error over all electrode sets and source eccentricities, for all simulations. In this respect, it is not equivalent to the tolerance measure of Soong et al., which also measures maximum error, but involves only one electrode set. Thus, our difference with Soong et al. may be due to using different criteria for rating the thin plate methods.

Like Soong et al., Perrin et al. (1987) also selected thin plate 3 as the best thin plate method. In their study, thin plate 3 had the lowest average RMS error of any thin plate method when interpolated from 19 electrodes. The average was taken over 65 dipole sources with varying orientations and eccentricities. The experiment of ours which was most comparable to theirs computed the mean global error interpolated from 26 electrodes, over six sets of five dipole sources each, at all eccentricities (see Fig.
4.2. Correlations of local and global parameters with interpolation error

We found that the single parameter having the highest linear correlation with error for all interpolation methods was potential gradient $g$. The local parameter $d$, distance to nearest electrode, had a weaker correlation. Curvature, the other local parameter associated with potential, had a very low correlation with error among splines, but a modest correlation for nearest neighbor. The global parameters source eccentricity ($E$) and number of electrodes ($N$) had modest correlations with error.

Thus, the error associated with potentials from high eccentricity sources is due more to the resulting high gradients than to high curvature. Although the correlation of $d$ was more modest than that of gradient, in every case the product of parameters $Edg$ was more highly correlated than gradient. And for fixed values of eccentricity, we found that $dg$ had higher correlation than gradient. Plots of local mean error versus $Edg$ showed a roughly linear relationship, going to 0 as $Edg$ goes to 0. Similarly, for fixed values of $E$, local mean error goes to 0 with $dg$.

This suggests that for a given source eccentricity, error can be controlled regardless of gradient size, by keeping $d$ sufficiently small. In fact, we simply need to keep $dg \leq K$, where the value of $K$ is determined by the appropriate functional relationship between local mean error and $dg$, given the known eccentricity, magnitude of dipole sources, and desired range of error. For example, if we are using 26 electrodes and we are interested in sources having unit dipole moment (as in our simulations) and high eccentricity, then in order to keep local mean error within a range of 0.05 $\mu$V, the top curve in Fig. 8 gives an estimated inequality $dg \leq 0.15$. Naturally, if our sources were known to have a different magnitude, then we would have to scale the curve in Fig. 8 appropriately to obtain the corresponding inequality.

Solving the general inequality gives $d \leq K/g$, hence the rule of thumb: to control interpolation error at a given head point, the distance to the nearest electrode must be on the order of the reciprocal of gradient. The higher the gradient at a point, the closer the point needs to be to an electrode for us to be confident of low interpolation error. Previous investigators (Gevins, 1990; Perrin et al., 1990) have noted the need for high electrode densities to adequately sample high spatial frequencies. Our results show that these densities need only be local in order to keep $d$ below a required threshold in the areas where high resolution is required. We illustrated this by resolving two closely spaced radial dipole sources at 88% eccentricity using a total of 47 electrodes, 24 of which were evenly spaced in a dense circular patch covering the sources (Fig. 12).

We also outlined and illustrated a method of predicting interpolation error based on a knowledge of the error distributions and the functional relationship of local mean error to $dg$. Our method showed some success at predicting one instance of interpolation error; but relying on the interpolated gradient values to estimate the true gradient values, it also underestimated interpolation error in another instance. Thus, reliable and accurate error prediction from the topographic map itself remains elusive. However, for interpolation of high eccentricity sources, our example did illustrate that any interpolated features far from an electrode in a topographic map should be viewed with skepticism. Thus, to aid in estimating the reliability of the interpolation at any location of a topographic map, the electrode locations should always be indicated on the map.

4.5. Summary of main conclusions

The following points summarize the main conclusions of this study:

1. Nearest neighbor and thin plate 4 interpolation should not be used if any of the other methods are available. The differences among spherical splines, thin plate 2 and thin plate 3 are relatively insignificant, and any of these methods may be used.

2. To control interpolation error, it is less important which method to use among the three best methods, than to know the possible potential gradients and adequately sample these by using appropriate electrode density. This density needs to be great enough only locally, in the areas of anticipated high potential gradients.

3. When interpolating data from deep sources, low gradients will be present and low numbers of electrodes may be used. When interpolating data from shallow
sources, high gradients will be present, but error can be controlled by locally varying the electrode density. The rule of thumb is that $d g \leq K$ must be maintained over the entire head. This means that at each head point, distance to nearest electrode must be proportional to the reciprocal of potential gradient.

4) When assessing the reliability of a topographic map, areas of high apparent gradient that are far from any electrode are likely to contain interpolation error. Moreover, for shallow sources, high interpolation error is possible in any area far from an electrode. In such a topographic map, all areas far from any electrode should be interpreted with caution. To aid in identifying these areas, electrode locations should be indicated on the map.

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References


